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On closure operator in quasi-minimal structures

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Abstract

Itai, Tsuboi and Wakai investigated the quasi-minimal structure [1]. They showed the geometric properties of quasi-minimal structures by using the countable closure. Here we discuss another closure operator in such structures.

1 Quasi-minimal structure and the countable closure

We recall some definitions.

Definition 1 An uncountable structure M is called *quasi−minimal* if every definable subset of M with parameters is at most countable or co-countable.

Let M be an uncountable structure and $A \subset M$. The *n−th countable closure* $\mathrm{ccl}_n^M(A)$ of A is inductively defined as follows:
$\mathrm{ccl}_0^M(A) = A$ and
$\mathrm{ccl}_{n+1}^M(A) = \bigcup\{\phi^M : \phi(x) \in L(\mathrm{ccl}_n^M(A)),\ \phi^M \text{ is countable}\}$
We put $\mathrm{ccl}^M(A) = \bigcup_{n\in\omega}\mathrm{ccl}_n^M$ (*the countable closure* of A). We omit the superscript M if it is clear from context.

And we recall the notion of pregeometry.

Definition 2 Let X be an infinite set and cl be a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of X. If the function cl satisfies the following properties, we say that (X, cl) is a *pregeometry*.
(I) $A \subseteq B \Longrightarrow A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,
(II) (Finite Character) $b \in \mathrm{cl}(A) \Longrightarrow b \in \mathrm{cl}(A_0)$ for some finite $A_0 \subseteq A$,
(III) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
(IV) (Exchange Axiom) $b \in \mathrm{cl}(A \cup \{c\}) - \mathrm{cl}(A) \Longrightarrow c \in \mathrm{cl}(A \cup \{b\})$.

The countable closure is a closure operator.

Fact 3 [1] *Let M be an uncountable quasi-minimal structure. Then it is clear that* (M, ccl) *satisfies the first three properties* (I) *through* (III).

The exchange axiom (IV) does not hold in general in (M, ccl). Itai et al. showed some conditions for M such that (M, ccl) satisfies the exchange axiom.

The notion of quasi-minimal structures is a generalization of minimal structures. And the countable closure corresponds to the algebraic closure naturally. Thus the countable closure is the canonical closure operator for quasi-minimal structures. It is easily checked that the countable closure of a set is either a definable set or a model in quasi-minimal structures. For further characterization, I defined some P-closures for them where P is a family of types. I considered that the countable closure is divided by some P-closures.

2 P-closures for quasi-minimal structures

First we recall some definitions from [2].

Definition 4 A family P of partial types is *A-invariant* if it is invariant under A-automorphisms. (where A is a subset of a sufficiently large saturated model as usual.)

Let P be an A-invariant family of partial types.

A partial type q over A is *P-internal* if for every realization a of q , there is $B \mathop{\smile\!\!\!\!\!\!\mid}_A a$, types $\bar{p}$ from P based on B, and realization $\bar{c}$ of $\bar{p}$, such that $a \in \mathrm{dcl}(B\bar{c})$.

A partial type q is *P-analysable* if for any $a \models q$, there are $(a_i : i < \alpha) \in \mathrm{dcl}(A,\ a)$ such that $tp(a_i/A,\ a_j\ :\ j < i)$ is P-internal for all $i < \alpha$, and $a \in \mathrm{bdd}(A,\ a_i : i < \alpha)$.

A complete type $q \in S(A)$ is *foreign* to P if for all $a \models q$, $B \mathop{\smile\!\!\!\!\!\!\mid}_A a$, and realizations $\bar{c}$ of extensions of types in P over B, we always have $a \mathop{\smile\!\!\!\!\!\!\mid}_{AB} \bar{c}$.

And let P be an $\emptyset$-invariant family of types.

A partial type q is *co-foreign* to P if every type in P is foreign to q.

The *P-closure* $\mathrm{cl}_P(A)$ of a set A is the collection of all element a such that $\mathrm{tp}(a/A)$ is P-analysable and co-foreign to P.

Remark 5 The P-analysable assumption could be modified or even omitted, resulting in a larger P-closure.

Fact 6 [2] *P-closure is a closure operator, i.e. it satisfies the axioms* (I) *and* (III) *in* Definition 2.

We recall another notion from [1] to define the family P of types in quasi-minimal structures.

Definition 7 Let M be quasi-minimal. Then a type $p(x)$ defined by
$$p(x) = \{\psi(x) \in L(M) \,:\, |\,\psi^M\,| \geqq \omega_1\}$$
is a complete type in $S(M)$. The type $p(x)$ is called the *main type* of M.

The family P of types should be defined such that the P-closure is included in the countable closure. So P is either the family of the restrictions of the main type or that of formulas which have uncountably many realizations in a quasi-minimal structure.

The notion of the main type is not elementary. It makes sense in a fixed quasi-minimal structure. Thus the P-closure must be defined in a fixed such structure. Otherwise it should be defined as the intersection between a fixed quasi-minimal model and the P-closure defined in the big model. We tried to define it in a fixed quasi-minimal structure at first. Thus we define the notion of foreignness and co-foreignness suitable for such P-closure.
For example,

Definition 8 Let M be a structure and $Th(M)$ be simple. And let P be a family of types over M.
An element a is *co-foreign* to P over A if for any $b \models p$ over B in P with $A \subseteq B$, and any $C \mathop{\smile\!\!\!\!\!\!|}_B b$, $b \mathop{\smile\!\!\!\!\!\!|}_{BC} a$ where all parameters are contained in M.
And we define $\mathrm{cl}^0_P(A) = \{a \in M \,:\, a \text{ is co-foreign to } P \text{ over } A.\}$

It is easily checked that the next facts hold.

Fact 9 *Let M be a quasi-minimal structure and $Th(M)$ be simple. And let P be the set of the restrictions of the main type closed under taking nonforking extensions. Then cl^0_P is a closure operator, i.e. (M, cl^0_P) satisfies the axioms* (I) *and* (III) *in* Definition 2.

Fact 10 *Under the same assumptions as the former fact. Then* $\mathrm{acl}(A) \subseteq \mathrm{cl}^0_P(A) \subseteq \mathrm{ccl}(A)$.

We need some modification of the definition so that cl^0_P satisfies the axiom (II) in Definition 2.

Next we must consider the definition of P-internality in quasi-minimal structures. We can define it in a fixed quasi-minimal structure as above.

But I can not fix definitions to realize the relation between the internality and the foreignness defined as above. It claims that the family P of types has some invariability under automorphisms.

And I can not construct quasi-minimal structures which have a proper P-closure yet.

References

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